ON UNIFORM ROTATIONS OF A BODY WITH A FIXED POINT

(O RAVNOMERNYKH VRASHCHENIIAKH TELA, IMEIUSHCHEGO NEPODVIZHNUIU TOCHKU)

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P.V.KHARLAMOV
(Novosibirsk)

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1. Initial equations. If a body is acted upon by the gravity forces and contains spinning masses (fly-wheels or liquid,circulating in multiply connected cavities), then its equations of motion are those shown in [1] which in the conventional notation [2] have the form

$$Ap^{\cdot} = (B - C)\,qr + \lambda_2 r - \lambda_3 q + e_2\gamma_3 - e_3\gamma_2, \qquad \gamma_1^{\cdot} = r\gamma_2 - q\gamma_3$$
$$(123,\ ABC,\ pqr)$$

or in vector notation

$$\mathbf{x}^{\cdot} = (\mathbf{x} + \boldsymbol{\lambda})\times\boldsymbol{\omega} + \mathbf{e}\times\boldsymbol{\gamma}, \qquad \boldsymbol{\gamma}^{\cdot} = \boldsymbol{\gamma}\times\boldsymbol{\omega} \tag{1.1}$$

Using the integrals

$$^1/_2\boldsymbol{\omega}\cdot\mathbf{x} - \mathbf{e}\cdot\boldsymbol{\gamma} = E, \qquad (\mathbf{x} + \boldsymbol{\lambda})\cdot\boldsymbol{\gamma} = k, \qquad \boldsymbol{\gamma}\cdot\boldsymbol{\gamma} = \Gamma^2 \tag{1.2}$$

we can easily transform them [3] into

$$\mathbf{e}\cdot[\mathbf{x}^{\cdot} + \boldsymbol{\omega}\times(\mathbf{x} + \boldsymbol{\lambda})] = 0 \tag{1.3}$$
$$[\mathbf{x}^{\cdot} + \boldsymbol{\omega}\times(\mathbf{x} + \boldsymbol{\lambda})]\cdot[\mathbf{e}\times(\mathbf{x} + \boldsymbol{\lambda})] + \mathbf{e}\cdot(\mathbf{x} + \boldsymbol{\lambda})\,(^1/_2\boldsymbol{\omega}\cdot\mathbf{x} - E) = k$$
$$[\mathbf{x}^{\cdot} + \boldsymbol{\omega}\times(\mathbf{x} + \boldsymbol{\lambda})]^2 + (^1/_2\,\boldsymbol{\omega}\cdot\mathbf{x} - E)^2 = \Gamma^2$$
$$\boldsymbol{\gamma} = (^1/_2\,\boldsymbol{\omega}\cdot\mathbf{x} - E)\,\mathbf{e} + [\mathbf{x}^{\cdot} + \boldsymbol{\omega}\times(\mathbf{x} + \boldsymbol{\lambda})]\times\mathbf{e}$$

2. The cone of the axes of the uniform rotation. Let the angular velocity vector be constant with respect to the body, $\boldsymbol{\omega}$ = const , then also $\mathbf{x}$ = const and from (1.3)

$$\mathbf{e}\cdot[\boldsymbol{\omega}\times(\mathbf{x} + \boldsymbol{\lambda})] = 0 \tag{2.1}$$

In Section 4 we shall investigate the case $\boldsymbol{\omega} = \omega\mathbf{e}$, while here we assume that

$$\boldsymbol{\omega} \neq \omega\mathbf{e} \tag{2.2}$$

hence, by (2.1)

$$\mathbf{x} + \boldsymbol{\lambda} = \alpha\boldsymbol{\omega} + \beta\mathbf{e} \tag{2.3}$$

where α and β are certain constants. Substituting (2.3) in (1.1), we obtain

$$\mathbf{e}\times\boldsymbol{\gamma} = \beta\,(\boldsymbol{\omega}\times\mathbf{e}), \qquad \text{or} \qquad \boldsymbol{\gamma} = \mu\mathbf{e} + \beta\,(\boldsymbol{\omega}\times\mathbf{e})\times\mathbf{e}$$

We define the constant μ from the last equation in (1.2) and obtain

$$\boldsymbol{\gamma} = (\boldsymbol{\omega}\times\mathbf{e})\times\mathbf{e} + \mathbf{e}\,\sqrt{\Gamma^2 - \beta^2\,[(\boldsymbol{\omega}\times\mathbf{e})\times\mathbf{e}]^2} \tag{2.4}$$

showing that the vector $\boldsymbol{\gamma}$ is also constant with respect to the body and by (1.1) $\boldsymbol{\gamma}\times\boldsymbol{\omega} = 0$. We conclude that the axis of rotation must be vertical

$$\boldsymbol{\gamma} = \frac{\Gamma}{\omega}\,\boldsymbol{\omega} \tag{2.5}$$

By (2.4) and (2.5) we have that

$$(\beta + \Gamma/\omega)\,\boldsymbol{\omega} = \{\beta\boldsymbol{\omega}\cdot\mathbf{e} + \sqrt{\Gamma^2 - \beta^2\,[(\boldsymbol{\omega}\times\mathbf{e})\times\mathbf{e}]^2}\}\,\mathbf{e}$$

By (2.2) this is only possible when $\beta = -\Gamma/\omega$; substituting this in (2.3), we obtain

$$\mathbf{x} + \boldsymbol{\lambda} = \alpha\boldsymbol{\omega} - (\Gamma/\omega)\,\mathbf{e}, \quad \text{or} \quad \boldsymbol{\omega}\times(\mathbf{x} + \boldsymbol{\lambda}) = (\Gamma/\omega)\,(\mathbf{e}\times\boldsymbol{\omega}) \tag{2.6}$$

Dot multiplying the last equation in (2.6) by $\boldsymbol{\lambda}$, we find

$$\boldsymbol{\lambda}\cdot(\boldsymbol{\omega}\times\mathbf{x}) = (\Gamma/\omega)\,\boldsymbol{\omega}\cdot(\boldsymbol{\lambda}\times\mathbf{e}) \tag{2.7}$$

In the coordinate system fixed in the body Equations (2.1) and (2.7) determine surfaces whose intersection is the directrix curve of the cone of axes of uniform rotation. When $\boldsymbol{\lambda} \to 0$ this cone becomes the cone $\mathbf{e}\cdot(\boldsymbol{\omega}\times\mathbf{x}) = 0$ (Staude[4]).

3. Equations of the cone in the principal coordinate axes. We shall write (2.1) and (2.7) in the form

$$(B - C)\,e_1qr + (C - A)\,e_2rp + (A - B)\,e_3pq + (\lambda_3e_2 - \lambda_2e_3)\,p + (\lambda_1e_3 - \lambda_3e_1)\,q + (\lambda_2e_1 - \lambda_1e_2)\,r = 0 \tag{3.1}$$

$$[(B - C)\,\lambda_1qr + (C - A)\,\lambda_2rp + (A - B)\,\lambda_3\,pq]\,\sqrt{p^2 + q^2 + r^2} + [(\lambda_2e_3 - \lambda_3e_2)\,p + (\lambda_2e_1 - \lambda_1e_3)\,q + (\lambda\,e_2 - \lambda_2e_1)\,r]\,\Gamma = 0 \tag{3.2}$$

Both these surfaces are at the origin tangent to the plane

$$(\lambda_3e_2 - \lambda_2e_3)\,p + (\lambda_1e_3 - \lambda_3e_1)\,q + (\lambda_2e_1 - \lambda_1e_2)r = 0 \tag{3.3}$$

The asymptotic cone of the surface (3.1) is the cone of Staude

$$(B - C)\,e_1qr + (C - A)\,e_2rp + (A - B)\,e_3pq = 0 \tag{3.4}$$

The asymptotic cone of the surface (3.2) is described by Equation

$$(B - C)\,\lambda_1qr + (C - A)\,\lambda_2rp + (A - B)\,\lambda_3\,pq = 0$$

The form of the surface (3.1) is determined by

$$I_1 = e_1e_2e_3\,(B - C)\,(C - A)\,(A - B), \qquad I_2 = \begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ e_1 & e_2 & e_3 \\ Ae_1 & Be_2 & Ce_3 \end{vmatrix} \tag{3.5}$$

When $I_1I_2 \neq 0$, then (3.1) becomes the equation of one sheet of a hyperboloid; in the case when $I_1 \neq 0$, $I_2 = 0$ becomes the equation of a cone; when $I_1 = 0$, but $I_2 \neq 0$, then (3.1) becomes the equation of a hyperbolic paraboloid, and finally when $I_1 = 0$ and $I_2 = 0$ then (3.1) determines two planes. In the case when $\lambda = \lambda e$ the plane (3.3) vanishes. In addition both these planes coincide with the cone (3.4).

Instead of (3.2) we can consider the surface

$$\frac{\omega}{\Gamma} = -\frac{(B - C)\,e_1qr + (C - A)\,e_2rp + (A - B)\,e_3pq}{(B - C)\,\lambda_1qr + (C - A)\,\lambda_2rp + (A - B)\,\lambda_3\,pq} \tag{3.6}$$

Let us consider the following method of investigating the cone of the axes of uniform rotation . We shall set $p = \omega\xi_1,\; q = \omega\xi_2,\; r = \omega\xi_3$. Obviously

$$\xi_1^2 + \xi_2^2 + \xi_3^2 = 1 \tag{3.7}$$

Equations (3.1) and (3.6) give

$$\omega = \frac{(\lambda_2 e_3 - \lambda_3 e_2)\,\xi_1 + (\lambda_3 e_1 - \lambda_1 e_3)\,\xi_2 + (\lambda_1 e_2 - \lambda_2 e_1)\,\xi_3}{(B-C)\,e_1\xi_2\xi_3 + (C-A)\,e_2\xi_3\xi_1 + (A-B)\,e_3\xi_1\xi_2} \tag{3.8}$$

$$\omega = -\Gamma \frac{(B-C)\,e_1\xi_2\xi_3 + (C-A)\,e_2\xi_3\xi_1 + (A-B)\,e_3\xi_1\xi_2}{(B-C)\,\lambda_1\xi_2\xi_3 + (C-A)\,\lambda_2\xi_3\xi_1 + (A-B)\,\lambda_3\xi_1\xi_2}$$

Hence

$$[(B-C)\,e_1\xi_2\xi_3 + (C-A)\,e_2\xi_3\xi_1 + (A-B)\,e_3\xi_1\xi_2]^2\Gamma + [(B-C)\,\lambda_1\xi_2\xi_3 + (C-A)\,\lambda_2\xi_3\xi_1 + (A-B)\,\lambda_3\xi_1\xi_2]\,[(\lambda_2 e_3 - \lambda_3 e_2)\,\xi_1 + (\lambda_3 e_1 - \lambda_1 e_3)\,\xi_2 + (\lambda_1 e_2 - \lambda_2 e_1)\,\xi_3] = 0 \tag{3.9}$$

Equation (3.9) determines on the unit sphere (3.7) the line of intersection of this sphere with the cone of the axes of uniform rotation (*).

We shall show a coordinate system which can be useful in the investigation of cones of the axes of permanent rotation. The unit vectors of this system are

$$\mathfrak{s}_1 = \frac{\mathbf{e} \times \boldsymbol{\lambda}}{\lambda}, \qquad \mathfrak{s}_2 = \mathbf{e}, \qquad \mathfrak{s}_3 = \frac{(\mathbf{e} \times \boldsymbol{\lambda}) \times \mathbf{e}}{\lambda}$$

Let $\varkappa$ be the angle between $\mathbf{e}$ and $\boldsymbol{\lambda}$, then

$$\boldsymbol{\lambda} = \lambda\,(\mathfrak{s}_2 \cos\varkappa + \mathfrak{s}_3 \sin\varkappa) \tag{3.10}$$

We shall substitute (3.10)and the vectors

$$\boldsymbol{\omega} = \omega_1\mathfrak{s}_1 + \omega_2\mathfrak{s}_2 + \omega_3\mathfrak{s}_3, \quad \mathbf{x} = x_1\mathfrak{s}_1 + x_2\mathfrak{s}_2 + x_3\mathfrak{s}_3$$

(besides $x_i = A_{i1}\omega_1 + A_{i2}\omega_2 + A_{i3}\omega_3$) in (2.1) and (2.7)

$$(A_{33} - A_{11})\,\omega_1\omega_3 + A_{31}\,(\omega_1^2 - \omega_3^2) + A_{23}\omega_1\omega_2 - A_{12}\omega_2\omega_3 - \lambda\omega_1 \sin\varkappa = 0$$

$$[(A_{22} - A_{11})\,\omega_1\omega_2 + A_{21}\,(\omega_1^2 - \omega_2^2) + A_{23}\omega_1\omega_3 - A_{12}\omega_2\omega_3 - \lambda\omega_1\cos\varkappa]\,\sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2} + \Gamma\omega_1 = 0$$

4. The special case $\boldsymbol{\omega} = \omega\mathbf{e}$. In this case by (1.1) we have

$$\boldsymbol{\gamma} = \alpha\mathbf{e} + \omega\,(\mathbf{x} + \boldsymbol{\lambda}) \tag{4.1}$$

The quantity α is determined from (1.2)

$$\alpha^2 + 2\alpha\omega\,\mathbf{e}\cdot(\mathbf{x} + \boldsymbol{\lambda}) + \omega^2\,(\mathbf{x} + \boldsymbol{\lambda})^2 = \Gamma^2$$

and is constant. Consequently the vector $\boldsymbol{\gamma}$ is a constant and in this case we have $\boldsymbol{\gamma} = \Gamma\mathbf{e}$. Hence by (4.1) we have $\mathbf{x} + \boldsymbol{\lambda} = \nu\mathbf{e}$, or in terms of the principal axes

$$\omega A e_1 - \nu e_1 + \lambda_1 = 0 \qquad (ABC,\ 123) \tag{4.2}$$

Eliminating ω and ν, we find that $I_2 = 0$, where I_2 is given by (3.5).

In the general case this condition means that a uniform rotation of a

*) For the case ω = const V.N.Drofa [5] introduced a redundant requirement that the coefficients of the corresponding equations should be proportional. Consequently, he concluded erroneously that a heavy gyrostat has in a general case only one axis of uniform rotation. A. Anchev [6 and 7] made the same mistake.

body about an axis through the center of gravity is possible under the condition that the vector λ is in the plane of the axis of rotation and of the normal to the ellipsoid of inertia passing through the point of intersection of the elipsoid with the axis of rotation. In particular, if the center of gravity is in one of the principal axes $e_2 = e_3 = 0$, then by (4.2) we have $\lambda_2 = \lambda_3 = 0$, which means that the vector λ should be directed along this principal axis. The magnitude of the angular velocity can be in this case completely arbitrary.

BIBLIOGRAPHY

1. Zhukovskii, N.E., O dvizhenii tverdogo tela, imeiushchego polosti napolnennye kapel'noi zhidkost'iu (On the motion of a rigid body with liquid-filled cavities). Complete Works, Vol.III, ONTI, 1936.
2. Kharlamov, P.V., Odno reshenie zadachi o dvizhenii tela, imeiushchego nepodvizhnuiu tochku (A solution for the motion of a body with a fixed point). *PMM* Vol.28, № 1, 1964.
3. Kharlamov, P.V., Ob uravneniiakh dvizheniia tiazhelogo tverdogo tela, imeiushchego nepodvizhnuiu tochku (On the equations of motion for a heavy body with a fixed point). *PMM* Vol.27, № 4, 1963.
4. Staude, O., Über permanente Rotationaxen bei der Bewegung eines schweren Körpers um einen festen Punkt. J.reine angew.Math., Vol.113, 1894.
5. Drofa, V.N., O permanentnykh osiakh dvizheniia tiazhelogo girostata okolo nepodvizhnoi tochki (On the permanent axes of motion for a heavy gyrostat near an immovable point). *PMM* Vol.25, № 5, 1961.
6. Anchev, A., Ob ustoichivosti permanentnykh vrashchenii tiazhelogo girostata (On the stability of permanent rotations of a heavy gyrostat). *PMM* Vol.26, № 1, 1962.
7. Anchev, A., V"rkhu permanentnite rotatsii na tezh"k zhirostat (On the permanent rotations of a heavy gyrostat).God.minno-geol.Inst.Sof., Vol.8, 1961-1962 (1963).

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